

of the Parts, and to order in what Direction and how much is necessary to be done.

III. *The Continuation of An Account of a Treatise of Fluxions, &c.* Book II. by Colin Mc Laurin, *Prof. Mathem.* Edinburgh. F. R. S. \*

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**I**N the first Book, the Author described the Method of Fluxions, and its Application to Problems of different Kinds, without making use of any particular Signs or Characters, by geometrical Demonstrations, that its Evidence might appear in the most simple and plain Form. In the second Book, he treats of the Method of Computation, or the Algebraic Part; to the Facility, Conciseness, and great Extent of which, the Improvements that have been made by this Method are in great measure to be ascribed. In order to obtain those Advantages, it was necessary to admit various Symbols into the Algebra: But the Number and Complication of those Signs must occasion some Obscurity in this Art, unless Care be taken to define their Use and Import clearly, with the Nature of the several Operations. An Example of this is given by an Illustration of one of the first Rules in Algebra. As it is the Nature of Quantity to be capable of Augmentation and Diminution, so Addition and Subtraction are the primary Operations in the Sciences that treat of it. The positive Sign implies an Incre-

\* See the Beginning of this Account, N<sup>o</sup> 468. p. 325.

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ment, or a Quantity to be added. The negative Sign implies a Decrement, or Quantity to be subtracted: And these serve to keep in our View what Elements enter into the Composition of Quantities, and in which manner, whether as Increments or Decrements. It is the same thing to subtract a Decrement as to add an equal Increment. As the Multiplication of a Quantity by a positive Number implies a repeated Addition of the Quantity, so the Multiplication by a negative Number implies a repeated Subtraction: And hence to multiply a negative Quantity, or Decrement, by a negative Number, is to subtract the Decrement as often as there are Units in this Number, and therefore is equivalent to adding the equal Increment the same Number of Times; or, when a negative Quantity is multiplied by a negative Number, the Product is positive. When we inquire into the Proportion of Lines in Geometry, we have no regard to their Position or Form; and there is no ground for imagining any other Proportion betwixt a positive and negative Quantity in Algebra, or betwixt an Increment and a Decrement, than that of the absolute Quantities or Numbers themselves. The Algebraic Expressions, however, are chiefly useful, as they serve to represent the Effects of the Operations; and such Expressions are not to be supposed equal that involve equal Quantities, unless the Operations denoted by the Signs are the same, or have the same Effect. Nor is such Expression to be supposed to represent a certain Quantity; for if the  $\sqrt{-1}$  should be said to represent a certain Quantity, it must be allowed to be imaginary, and yet to have a real Square; a way of speaking which it is better to avoid. It denotes only, that

that an Operation is supposed to be performed on the Quantity that is under the radical Sign. The Operation is indeed in this Case imaginary, or cannot succeed; but the Quantity that is under the radical Sign, is not less real on that Account. The Author mentions those things briefly, because they belong rather to a Treatise of Algebra than of Fluxions, wherein the common Algebra is admitted.

In order to avoid the frequent Repetition of figurative Expressions in the Algebraic Part, the Fluxions of Quantities are here defined to be any Measures of their respective Rates of Increase or Decrease, while they are supposed to vary (or flow) together. These may be determined by comparing the Velocities of Points that always describe Lines proportional to the Quantities, as in the First Book; but they may be likewise determined, without having recourse to such Suppositions, by a just Reasoning from the simultaneous Increments or Decrements themselves. While the Quantity  $A$  increases by Differences equal to  $a$ ,  $2A$  increases by Differences equal to  $2a$ , and (supposing  $m$  and  $n$  to be invariable)  $\frac{mA}{n}$  increases by Differences equal to  $\frac{ma}{n}$ , and therefore at a greater or less Rate than  $a$ , in proportion as  $m$  is greater or less than  $n$ . Thus a Quantity may be always assigned that shall increase at a greater or less Rate than  $A$ , (*i. e.* shall have its Fluxion greater or less than the Fluxion of  $A$ ) in any Proportion; and a Scale of Fluxions may be easily conceived, by which the Fluxions of any other Quantities of the same kind may be measured.

Let

Let  $B$  be any other Quantity whose relation to  $A$  can be expressed by any Algebraic Form; and while  $A$  increases by equal successive Differences, suppose  $B$  to increase by Differences that are always varying. In this Case,  $B$  cannot be supposed to increase at any one constant Rate; but it is evident, that if  $B$  increase by Differences that are always greater than the equal successive Differences by which  $\frac{mA}{n}$  increases at the same time, then  $B$  cannot be said to increase at a less Rate than  $\frac{mA}{n}$ ; or if the Fluxion of  $A$  be represented by  $a$ , the Fluxion of  $B$  cannot be less than  $\frac{ma}{n}$ . And if the successive Differences of  $B$  be always less than those of  $\frac{mA}{n}$ , then surely  $B$  cannot be said to increase at a greater Rate than  $\frac{mA}{n}$ ; or the Fluxion of  $B$  cannot be said to be greater in this Case than  $\frac{ma}{n}$ .

From those Principles the primary Propositions in the Method of Fluxions, and the Rules of the direct Method, with the fundamental Rules of the inverse Method, are demonstrated. We must be brief in our Account of the Remainder of this Book. The Rule for finding the Fluxion of a Power is not deduced, as usually, from the Binomial Theorem, but from one that admits of a much easier Demonstration from the first Algebraic Elements, *viz.* That when  $n$  is any integer positive Number, if the Terms  $E^{n-1}$ ,  $E^{n-2}F$ ,  $E^{n-3}F^2$ ,  $E^{n-4}F^3$ , . . . .  $F^{n-1}$ , (wherein the Index of  $E$  constantly decreases, and that of  $F$  increases by the same Difference Unit) be multiplied by  $E-F$ , the Sum of the Products is  $E^n - F^n$ ; from which it is obvious, that when  $E$  is greater than  $F$ ,  
then

then  $E^n - F^n$  is less than  $nE^{n-1} \times \overline{E-F}$  but greater than  $nF^{n-1} \times \overline{E-F}$ .

The Ruels are sometimes proposed in a Form somewhat different from the usual manner of describing them, with a View to facilitate the Computations both in the direct and inverse Method. Thus, when a Fraction is proposed, and the Numerator and Denominator are resolved into any Factors, it is demonstrated, that the Fluxion of the Fraction divided by the Fraction is equal to the Sum of the Quotients, when the Fluxion of each Factor of the Numerator is divided by the Factor itself, diminished by the Quotients that arise by dividing in like manner the Fluxion of each Factor of the Denominator by the Factor.

The Notation of Fluxions is described in Chap. 2. with the Rules of the direct Method, and the fundamental Rules of the inverse Method. The latter are comprehended in Seven Propositions, Six of which relate to Fluents that are assignable in finite Algebraic Terms, and the Seventh to such as are assigned by infinite Series. It is in this Place the Author treats of the Binomial and Multinomial Theorems (because of their Use on this Occasion), and they are investigated by the direct Method of Fluxions. The same Method is applied for demonstrating other Theorems, by which an Ordinate of a Figure being given, and its Fluxions determined, any other Ordinate and *Area* of the Figure may be computed. The most useful Examples are described in this Chapter, by computing the Series's that serve for determining the Arc from its Sine or Tangent, and the

Logarithm from its Number, and conversely the Sine, Tangent, or Secant, from the Arc, and the Number from its Logarithm.

The inverse Method is prosecuted farther in the Third Chapter, by reducing Fluents to others of a more simple Form, when they are not assignable by a finite Number of Algebraic Terms. When a Fluent can be assigned by the Quadrature of the Conic Sections, (and consequently by circular Arcs or Logarithms) this is considered as the second Degree of Resolution; and this Subject is treated at Length. An Illustration is premised of the Analogy betwixt Elliptic and Hyperbolic Sectors formed by Rays drawn from the Centres of the Figures: The Properties of the latter are sometimes more easily discovered because of their Relation to Logarithms, and lead us in a brief manner to the analogous Properties of Elliptic Sectors, and particularly to some general Theorems concerning the Multiplication and Division of circular Sectors or Arcs. When Two Points are assumed in an Hyperbola, and also in an Ellipsis, so that the Sectors terminated by the Semi-axis; and the Two Semi-diameters, belonging to those Points, are in the same given *Ratio* in both Figures, then the Relation betwixt the Semi-axis and the Two Ordinates drawn from those Points to the other Axis, is always defined by the same, or by a similar Equation in both Figures. This Proposition serves for demonstrating Mr. *Cotes's* celebrated Theorem, as it is extended by M. *De Moivre*, by which a Binomial or Trinomial is resolved into its quadratic Divisors, and various Fluents are reduced to circular Arcs and Logarithms. The Demonstrations are also rendered more easy of  
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the Theorems concerning the Resolution of a Fraction, that has a multinomial Denominator, into Fractions that have the simple or quadratic Divisors of the Multinomial for their several Divisors. These Demonstrations are derived from the Method of Fluxions itself, without any foreign Aid; or invariable Coefficients are determined by supposing the variable Quantity or its Fluxions to vanish.

When a Fluent cannot be assigned by the Arcs of Conic Sections, it may however be measured by their Arcs in some Cases; and this may be considered as the Third Degree of Resolution, or the Fluents may be called of the Third Order. On this Occasion some Fluents are found to depend on the Rectification of the Hyperbola and Ellipsis, which have been formerly esteemed of an higher kind. The Construction of the elastic Curve, with its Rectification, and the Measure of the Time of Descent in an Arch of a Circle, are derived from Hyperbolic and Elliptic Arcs; and the Fluents of this kind are compared with those of the First or Second Order by infinite Series. Because there are Fluents of higher kinds than these, the Trajectories above-mentioned, which are described by a centripetal Force, that is, as some Power of the Distance from a given Centre, when the Velocity of the Projection is that which would be acquired by an infinite Descent, or by such a centrifugal Force, and the Velocity is such as would be acquired by flying from the Centre, are employed for representing them. A simple Construction of these Trajectories had been given above, by drawing Rays from the Centre to a Right Line given in Position, increasing or diminishing the Logarithms of those Rays always in a given

*Ratio*, and increasing or diminishing the Angles contained by them and the Perpendicular in the same *Ratio*. From any Figure of this kind a Series of Figures is derived by determining the Intersections of the Tangents of the Figure with the Perpendiculars from the Centre. Every Series of this kind gives Two distinct sort of Fluents; and any one Fluent being given, all the other Fluents taken alternately from it in the Series depend upon it, or are measured by it; but it does not appear, that the Fluents of one sort can be compared with those of the other sort, or with those of any different Series of this kind.

The inverse Method is prosecuted farther in the 4th Chapter, by various Theorems concerning the Area when the Ordinate is expressed by a Fluent, or when the Ordinate and Base are both expressed by Fluents. The First is the XIth Proposition of Sir *Isaac Newton's* Treatise of Quadratures. In Art. 819, 820, &c. the Author supposes the Ordinate and Base to be both expressed by Fluents, and shews, in many Cases, that the Area may be assigned by the Product of Two simple Fluents, as of Two circular Arcs, or of a circular Arc and a Logarithm. This Subject deserves to be prosecuted, because the Resolution of Problems is rendered more accurate and simple, by reducing Fluents to the Products of Fluents already known, than by having immediately recourse to infinite Series. One of the Examples in Art. 822. may be easily applied for demonstrating, that the Sum of the Fractions which have Unit for their common Numerator, and the Squares of the Numbers 1, 2, 3, 4, 5, 6, &c. in their natural Order,

for



for their successive Denominators, is One-sixth Part of the Number, which expresses the *Ratio* of the Square of the Periphery of a Circle to the Square of its Diameter; which is deduced by Mr. *Euler*, *Comment. Petrophol.* Tom. 7. in a different manner; and other Theorems of this kind may be demonstrated from the same or like Principles.

The Series that is deduced by the usual Methods for computing the Area or Fluent, converge in some Cases at so slow a Rate, as to be of little or no Use without some farther Artifice. For Example: The Sum of the first Thousand Terms of Lord *Brounker's* Series for the Logarithm of 2, is deficient in the fifth Decimal. In order therefore to render the Account of the inverse Method more complete, the Author shews how this may be remedied, in many Cases, by Theorems derived from the Method of Fluxions itself, which likewise serve for approximating readily to the Values of Progressions, and for resolving Problems that are commonly referred to other Methods. Those Theorems had been described in the First Book, Art. 352, &c. but the Demonstration and Examples were referred to this Place, as requiring a good deal of Computation. The Base being supposed equal to Unit, and its Fluxion also equal to Unit, let half the Sum of the extreme Ordinates be represented by *a*, the Difference of the first Fluxions of these Ordinates by *b*, the Difference of their Third, Fifth, Seventh and higher alternate Fluxions by *c*, *d*, *e*, &c. then the Area shall be equal to  $a - \frac{b}{12} + \frac{c}{720} - \frac{d}{30240} + \frac{e}{1209600} - \dots$ , &c. which is the first Theorem for finding the Area. The rest remaining, let

let  $a$  now represent the middle Ordinate, and the Area shall be equal  $a + \frac{b}{24} - \frac{7c}{5760} + \frac{31d}{967680} - \frac{127e}{154828800} +, \&c.$

And this is the Theorem which the Author makes most Use of. When the several intermediate Ordinates represent the Terms of a Progression, the Area is computed from their Sum, or conversely their Sum is derived from the Area, by Theorems that easily flow from these.

These general Theorems are afterwards applied for finding the Sums of the Powers of any Terms in Arithmetical Progression, whether the Exponents of the Powers be Positive or Negative, and for finding the Sums of their Logarithm, and thereby determining the *Ratio* of the *Uncia* of the middle Term of a Binomial of a very high Power to the Sum of all the *Unciæ*. This last Problem was celebrated amongst Mathematicians some Years ago, and by endeavouring to resolve it by the Method of Fluxions the Author found those Theorems, which give the same Conclusions that are derived from other Methods. They are likewise applied for computing *Areas* nearly from a few equidistant Ordinates, and for interpolating the intermediate Terms of a Series, when the Nature of the Figure can be determined, whose Ordinates are as the Differences of the Terms.

In the last Chapter, the general Rules, derived from the Method of Fluxions for the Resolution of Problems, are described and illustrated by Examples. After the common Theorems concerning Tangents, the Rules for determining the greatest and least Ordinates, with the Points of contrary Flexure, and the Precautions that are necessary to render them accurate and

and general, (which were described above) are again demonstrated. Next follow the Algebraic Rules for finding the Centre of Curvature, and determining the Caustics by Reflexion and Refraction, and the centripetal Forces. The Construction of the Trajectory is given, which is described by a Force that is inversely as the Fifth Power of the Distance from the Centre, because this Construction requires Hyperbolic and Elliptic Arcs, and because a remarkable Circumstance takes place in this Case, (and indeed in an Infinity of other Cases) which could not obtain in those that have been already constructed by others, *viz.* That a Body may continually descend in a spiral Line towards the Centre, and yet never approach so near to it as to descend to a Circle of a certain *Radius*; and a Body may recede for ever from the Centre, and yet never arise to a certain finite Altitude. The Construction of the Cases wherein this obtains is performed by Logarithms or Hyperbolic Areas, the Angles described about the Centre being always proportional to the Hyperbolic Sectors, while the Distances from the Centre are directly or inversely as the Tangents of the Hyperbola at its Vertex. The Circle is an Asymptote to the Spiral; and this can never be, unless the Velocities requisite to carry Bodies in Circles increase while the Distances decrease, (or decrease while the Distances increase) in a higher Proportion than the Velocity in the Trajectory; that is, unless the Force be inversely as a higher Power of the Distance than the Cube. Next follow Theorems for computing the Time of Descent in any Arc of a Curve, for finding the Resistance and Density of the Medium when the Trajectory and centripetal Force

Force are given, and for defining the *Catenaria* and Line of swiftest Descent in any Hypothesis of Gravity.

Then the usual Rules are derived from the inverse Method for computing the Area, the Solid generated by it, the Arc of the Curve, and the Surface described by it revolving about a given Axis. The meridional Parts in a Sphere, and any Spheroid, are determined with the same Accuracy, and almost equal Facility. The Attraction of a Spheroid at the Equator, as well as at the Poles, is determined in a more general manner than in the First Book, or in a Piece of the Author's published at *Paris* in 1740. which obtained a Part of the Prize proposed by the *Royal Academy of Sciences* for that Year. Several Mechanical Problems are resolved, concerning the Proportion the Power ought to bear to the Weight, that the Engine may produce the greatest Effect in a given Time; and concerning the most advantageous Position of a Plane which moves parallel to itself, that a Stream of Air or Water may impel it with the greatest Force, having regard to the Velocity which the Plane may have already acquired. On this Occasion, it is shewn, that the Wind ought to strike the Sails of a Wind-mill in a greater Angle than that of  $54^{\circ} 44'$ , against what has been deduced from the same Principles by a learned Author. The same Theory is applied to the Motion of Ships, abstracting from the Lee-way, but having regard to the Velocity of the Ship; and amongst other Conclusions it appears, that the Velocity of a Vessel of one Sail may be greater with a Side-wind, than when she sails directly before the Wind; which, perhaps, may be the Case of those seen by Captain *Dampier*

*Dampier* in the *Ladron Islands*, that sailed at the Rate of 12 Miles in half an Hour with a Side-wind.

The Remainder of this Chapter is employed in reducing Equations from second to first Fluxions; constructing the elastic Curve by the Rectification of the equilateral Hyperbola; determining the Vibrations of Musical Chords; resolving Problems concerning the *Maxima* and *Minima*, that are proposed with Limitations, relating to the Perimeter of the Figure, its Area, the Solid generated by this Area, &c. with Examples of this kind concerning the Solid of least Resistance; and concludes with an Instance of the Theorems by which the Value of the Ordinate may be determined from the Value of the Area, by common Algebra, and by observing, that it is not absolute, but relative Space and Motion, that is supposed in the Method of Fluxions.